3. Appendix Inverses of Arithmetic functions

A result stated and not proved in the lectures was

Theorem 3.40 *An arithmetic function f has an inverse under $*$ if, and only if, $f(1) \neq 0$.*

Proof ($\Rightarrow$) Assume f has an inverse, g say, so $g * f = \delta$. In particular $g * f(1) = \delta(1) = 1$, i.e.

$$g(1)\, f(1) = 1. \tag{15}$$

Hence $f(1) \neq 0$.

($\Leftarrow$) Assume that $f(1) \neq 0$. Define g inductively.

So start with $g(1)\, f(1) = 1$, i.e. $g(1) = 1/f(1)$.

Assume that $g(n)$ has been defined for all $1 \leq n \leq k$. Define $g(k+1)$ to ensure that

$$\sum_{ab=k+1} g(a)\, f(b) = 0, \quad \text{i.e.} \quad g(k+1)\, f(1) = -\sum_{\substack{ab=k+1 \\ a \neq k+1}} g(a)\, f(b)\,. \tag{16}$$

This definition makes sense. In the sum on the right hand side we have $ab = k+1$ and $a \neq k+1$, in which case $a \leq k$ and we have assumed g has already been defined on these a and the values can be fed in to give the definition of $g(k+1)$.

In this way the infinite sequence of equalities

$$\begin{aligned} g * f(1) &= \delta(1) = 1, \\ g * f(2) &= \delta(2) = 0, \\ g * f(3) &= \delta(3) = 0, \\ &\vdots \end{aligned}$$

are satisfied. Thus

$$\sum_{ab=n} g(a)\, f(b) = \delta(n)\,,$$

for all $n \geq 1$ which means $g * f = \delta$. ■

We can go further and ask, if f is multiplicative and $f(1) \neq 0$ is f^{-1} multiplicative?

Theorem 3.41 *If f is multiplicative and has an inverse f^{-1} then the inverse is multiplicative.*

Proof Assume that f is multiplicative. Then $f(1) = 1 \neq 0$ and so by the previous theorem f has an inverse, defined iteratively by $f^{-1}(1) = 1$ and

$$f^{-1}(N) = -\sum_{\substack{d|N\\ d\neq N}} f^{-1}(d)\, f\left(\frac{N}{d}\right), \tag{17}$$

for all $N \geq 2$.

We require to show that $f^{-1}(m_1m_2) = f^{-1}(m_1)\ f^{-1}(m_2)$ for all coprime pairs (m_1, m_2). The proof is by induction on m_1m_2.

The base case is the coprime pair (m_1, m_2) with $m_1m_2 = 1$. This is just $m_1 = m_2 = 1$. From its definition we have $f^{-1}(1) = 1$ in which case $f^{-1}(1) = 1 = f^{-1}(1)\ f^{-1}(1)$ and so the result holds in this case.

Assume that $f^{-1}(m_1m_2) = f^{-1}(m_1)\ f^{-1}(m_2)$ for all coprime pairs with $m_1m_2 \leq k$, for some $k \geq 2$.

Let (n_1, n_2) be a coprime pair with $n_1n_2 = k+1$. Apply (17) with $N = n_1n_2$ to get

$$f^{-1}(n_1n_2) = -\sum_{\substack{d|n_1n_2\\ d\neq n_1n_2}} f^{-1}(d)\, f\left(\frac{n_1n_2}{d}\right).$$

Because $\gcd(n_1, n_2) = 1$, there is a one-to-one map between the divisors d of n_1n_2 and the pairs of divisors (d_1, d_2) with $d_1|n_1$ and $d_2|n_2$. Thus

$$f^{-1}(n_1n_2) = -\sum_{\substack{d_1|n_1\ d_2|n_2\\ d_1d_2\neq n_1n_2}} f^{-1}(d_1d_2)\, f\left(\frac{n_1}{d_1}\frac{n_2}{d_2}\right)$$

In this sum $d_1d_2 \neq n_1n_2$ and so $d_1d_2 < n_1n_2$, i.e. $d_1d_2 \leq k$. By the inductive hypothesis $f^{-1}(d_1d_2) = f^{-1}(d_1)\, f^{-1}(d_1)$. Hence

$$\begin{aligned} f^{-1}(n_1n_2) &= -\sum_{\substack{d_1|n_1\ d_2|n_2\\ d_1d_2\neq n_1n_2}} f^{-1}(d_1)\, f^{-1}(d_2)\, f\left(\frac{n_1}{d_1}\right) f\left(\frac{n_2}{d_2}\right) \\ &= -\sum_{\substack{d_1|n_1\ d_2|n_2\\ d_1\neq n_1, d_2\neq n_2}} \ldots - \sum_{\substack{d_1|n_1\ d_2|n_2\\ d_1=n_1, d_2\neq n_2}} \ldots - \sum_{\substack{d_1|n_1\ d_2|n_2\\ d_1\neq n_1, d_2=n_2}} \ldots \end{aligned} \tag{18}$$

The first term here equals

$$-\left(-\sum_{\substack{d_1|n_1\\d_1\neq n_1}} f^{-1}(d_1)\, f\left(\frac{n_1}{d_1}\right)\right)\left(-\sum_{\substack{d_2|n_2\\d_2\neq n_2}} f^{-1}(d_2)\, f\left(\frac{n_2}{d_2}\right)\right)$$

$$= -f^{-1}(n_1)\, f^{-1}(n_2)$$

by (17) applied twice with $N = n_1$ and $N = n_2$. The second term in (18) equals

$$\sum_{\substack{d_1|n_1\ d_2|n_2\\d_1=n_1,d_2\neq n_2}} f^{-1}(d_1)\, f^{-1}(d_2)\, f\left(\frac{n_1}{d_1}\right) f\left(\frac{n_2}{d_2}\right)$$

$$= \sum_{\substack{d_2|n_2\\d_2\neq n_2}} f^{-1}(n_1)\, f^{-1}(d_2)\, f(1)\, f\left(\frac{n_2}{d_2}\right)$$

$$= -f^{-1}(n_1)\, f^{-1}(n_2)\,,$$

by (17) applied with $N = n_2$. And the same result holds for the third term in (18). Thus

$$\begin{aligned} f^{-1}(n_1 n_2) &= -f^{-1}(n_1)\, f^{-1}(n_2) + f^{-1}(n_1)\, f^{-1}(n_2) + f^{-1}(n_1)\, f^{-1}(n_2) \\ &= f^{-1}(n_1)\, f^{-1}(n_2)\,. \end{aligned}$$

So the result holds for all coprime pairs with product $k+1$. Hence, by induction, the result holds for all coprime pairs, i.e. f^{-1} is multiplicative. ■